Chapter 4
Asymptotics

(A) CLT

Suppose $Y_1, \ldots, Y_n$ **iid** with $\mathbb{E}\{Y_i\} = \mu$
& $\text{var}\{Y_i\} = \sigma^2$
$i = 1, \ldots, n$

moments

The **central limit theorem** (CLT) says

$$\frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y}_n \sim N(\mu, \sigma^2/n) \quad \text{as } n \to \infty$$

Standardization

$$Z \equiv \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \quad \text{as } n \to \infty$$

Consider our coin flipping / Bernoulli inference problem (is the coin fair?)

$$Y_i \overset{iid}{\sim} \text{Bern}(\theta) \qquad \mathbb{E}\{Y_i\} = \theta \leftarrow \mu$$

$$\text{Var}\{Y_i\} = \theta(1-\theta) \leftarrow \underbrace{\sigma^2}_{\text{CLT}}$$

By CLT $\underset{\substack{\uparrow \\ \text{stat}}}{\bar{Y}_n} \sim N\left(\theta, \underbrace{\theta(1-\theta)/n}_{\text{squared SE}}\right)$

$$H_0: \theta = 1/2$$
$$H_1: \theta \neq 1/2$$

$$\frac{\bar{Y}_n - \overset{\swarrow 1/2}{\theta}}{\sqrt{\underbrace{\theta(1-\theta)}/n}}$$

The <u>weak law of large numbers</u> (WLLN)

says $\bar{Y}_n \to \mu \ (\equiv \theta)$ as $n \to \infty$

Therefore $\bar{Y}_n \sim N(\theta, \bar{Y}_n(1-\bar{Y}_n)/n)$

$$\frac{\bar{Y}_n - \theta}{\sqrt{\bar{Y}_n(1-\bar{Y}_n)n}} \sim N(0,1) \quad \text{as } n \to \infty$$

Chapter 4
Continued

(B) Asymptotic distribution of the MLE (AMLE)

Let θ be a p-dim parameter to a family of distributions emitting a log likelihood

$$\ell(\theta; y_1, \dots, y_n)$$

e.g. $\theta = (\mu, \sigma^2)$ in N
or $\underline{\theta \text{ prop in } Bern(\theta)}$
$\downarrow$
$p = 1$

Let $\hat{\theta}$ be MLE, which you might get by solving the <u>likelihood equations</u>

$$\left.\frac{\partial \ell}{\partial \theta_j}\right|_{\hat{\theta}_n} \overset{set}{=} \phi \quad \& \text{ solve } j = 1, \dots, p$$

Then

$$\hat{\theta}_n (Y_1, \ldots, Y_n) \equiv \hat{\theta}_n \sim N(\theta, 1/n \cdot i_1^{-1}(\theta))$$

as $n \to \infty$

where $i(\theta) = -\mathbb{E}\{\underbrace{\nabla \nabla^T}\, l(\theta, Y)\}$

<u>Fisher Information</u> $\qquad l''(\theta)$ for $p=1$

$$\downarrow$$

$$i(\theta) \equiv i_n(\theta) \overset{iid}{=} n\, i_1(\theta)$$

<u>For example</u> $\qquad Y_i \overset{iid}{\sim} Bern(\theta) \quad i=1,\ldots,n$

$$L(\theta; y_1, \ldots, y_n) = \prod_{i=1}^{n} \theta^{y_i} (1-\theta)^{1-y_i}$$

$$= \theta^{\sum y_i} (1-\theta)^{n - \sum y_i}$$

$$l(\theta) = (\sum y_i) \log \theta + (n - \sum y_i) \log (1-\theta)$$

$$\phi \overset{set}{=} l'(\hat{\theta}) = \frac{\sum y_i}{\hat{\theta}} + \frac{n - \sum y_i}{1 - \hat{\theta}} (-1)$$

$$\boxed{\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} Y_i}$$

Now, to calculate $i_1(\theta)$

$$l(\theta, y_1) = y_1 \log\theta + (1-y_1)\log(1-\theta)$$

$$l'(\theta, y_1) = \frac{y_1}{\theta} + \frac{(1-y_1)}{1-\theta}(-1)$$

$$l''(\theta, y_1) = -\frac{y_1}{\theta^2} - \frac{(1-y_1)}{(1-\theta)^2}\cancel{(-1)}$$

$$-\mathbb{E}\{l''(\theta, Y_1)\} = \frac{\cancel{\theta}}{\theta^{\cancel{2}}} + \frac{\cancel{(1-\theta)}}{(1-\theta)^{\cancel{2}}}$$

$\uparrow$

$$= \frac{1}{\theta} + \frac{1}{1-\theta}$$

so $\boxed{i_1(\theta) = \frac{1}{\theta(1-\theta)}}$

Therefore AMLE $\hat{\theta}_n \sim N(\theta, \theta(1-\theta)/n)$

this is the same result as from CLT